

*-products on quantum spaces

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Abstract. In this paper we present explicit formulas for the *-product on quantum spaces which are of particular importance in physics, i.e., the q -deformed Minkowski space and the q -deformed Euclidean space in 3 and 4 dimensions, respectively. Our formulas are complete and formulated using the deformation parameter q . In addition, we worked out an expansion in powers of $h = \ln q$ up to second order, for all considered cases.

1 Introduction

Non-commutative space-time structures seem to be one of the most hopeful notions in formulating finite quantum field theories [1]. Even in the content of string theory non-commutative geometries have recently been studied [2,3]. Especially quantum spaces which can lead to a lattice-like space-time structure provide a natural frame work for a realistic non-commutative field theory [4,5]. In order to do so we employ the *-product formalism which represents the non-commutative structure on a commutative one [6–8].

In the following we want to concern ourselves with coordinates which have quantum groups as their underlying symmetry structure, in very much the same way as for example the classical Minkowski space has the Lorentz group as its underlying symmetry structure. Quantum groups are q -deformations of function algebras over classical Lie groups (or q -deformations of the enveloping algebra of classical Lie algebras respectively) [9]. The algebra generated by the coordinates is a comodule algebra of some quantum group and is called a quantum space. So we can define the coordinate algebra \mathcal{A}_q generated by the coordinates $\hat{X}_1, \hat{X}_2, \dots, \hat{X}_n$ as

$$\mathcal{A}_q = \frac{\mathbb{C}\langle \hat{X}_1, \hat{X}_2, \dots, \hat{X}_n \rangle}{\mathcal{R}}, \quad (1.1)$$

where the relations between these coordinates reflect the quantum symmetry and therefore determine the ideal \mathcal{R} . Formal power series in the coordinates are allowed in \mathcal{A}_q .

The algebra \mathcal{A}_q satisfies the Poincaré-Birkhoff-Witt property, i.e., the dimension of the subspace spanned by monomials of a fixed degree is the same as the dimension of the subspace spanned by monomials in commutative variables of the same degree. Taking this property and choos-

ing the monomials of normal ordering $\hat{X}_1^{i_1} \hat{X}_2^{i_2} \dots \hat{X}_n^{i_n}$ as basis of \mathcal{A}_q , we can establish an isomorphism between \mathcal{A}_q and the commutative algebra \mathcal{A} generated by ordinary coordinates x_1, x_2, \dots, x_n , as vector spaces.

$$W : \mathcal{A} \longrightarrow \mathcal{A}_q \quad (1.2)$$

$$W(x_1^{i_1} \dots x_n^{i_n}) = \hat{X}_1^{i_1} \dots \hat{X}_n^{i_n}.$$

Let us consider a formal power series in the algebra \mathcal{A}_q , $\hat{f} = \sum_i a_{i_1 \dots i_n} \hat{X}_1^{i_1} \dots \hat{X}_n^{i_n}$, the image under W is $f = \sum_i a_{i_1 \dots i_n} x_1^{i_1} \dots x_n^{i_n}$, with the same coefficients $a_{i_1 \dots i_n}$. This isomorphism of vector spaces can be extended to an isomorphism of algebras introducing a non-commutative product in \mathcal{A} , the so-called *-product. This product is defined by the relation

$$W(f * g) = W(f)W(g), \quad (1.3)$$

where f and g are formal power series in \mathcal{A} . This will give the usual multiplication in the limit $q \rightarrow 1$ [10,11,22]. In order to do field theory on non-commutative spaces one needs to have a notion of integration. This notion is given in [12]. In this paper we will work out the necessary *-product for some relevant quantum spaces, namely for the q -deformed 3- and 4-dimensional Euclidean space and the q -deformed Minkowski space.

2 q -deformed 3-dimensional Euclidean space

The algebra of the q -deformed version of 3-dimensional Euclidean space is the algebra generated by the coordinates $\hat{X}_3, \hat{X}_+, \hat{X}_-$, satisfying the following relations [4, 13]

$$\hat{X}_3 \hat{X}_+ = q^2 \hat{X}_+ \hat{X}_3, \quad \hat{X}_- \hat{X}_3 = q^2 \hat{X}_3 \hat{X}_-, \quad (2.4)$$

$$\hat{X}_- \hat{X}_+ = \hat{X}_+ \hat{X}_- + \lambda \hat{X}_3 \hat{X}_3,$$

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where $\lambda = q - q^{-1}$. q is the complex deformation parameter.

As a basis of this quantum space we can take the monomials of normal ordering, $\hat{X}_+^{n_+} \hat{X}_3^{n_3} \hat{X}_-^{n_-}$. Using the isomorphism W introduced before, we can assign each monomial in commutative coordinates a normal ordered expression of non-commutative coordinates,

$$W(x_+^{n_+} x_3^{n_3} x_-^{n_-}) = \hat{X}_+^{n_+} \hat{X}_3^{n_3} \hat{X}_-^{n_-}. \tag{2.5}$$

The *-product on monomials is then defined by the condition

$$W((x_+^{n_+} x_3^{n_3} x_-^{n_-}) * (x_+^{m_+} x_3^{m_3} x_-^{m_-})) = W(x_+^{n_+} x_3^{n_3} x_-^{n_-}) W(x_+^{m_+} x_3^{m_3} x_-^{m_-}) \tag{2.6}$$

The right hand side of (2.6) has to be rewritten in normal ordering, using relations (2.4). For this aim, we need to calculate the commutation relations for $\hat{X}_-^{n_-} \hat{X}_3^{m_3}$, $\hat{X}_3^{n_3} \hat{X}_+^{m_+}$ and $\hat{X}_-^{n_-} \hat{X}_+^{m_+}$.

These commutation relations read

$$\hat{X}_-^{n_-} \hat{X}_3^{m_3} = q^{2n_- m_3} \hat{X}_3^{m_3} \hat{X}_-^{n_-}, \tag{2.7}$$

$$\hat{X}_3^{n_3} \hat{X}_+^{m_+} = q^{2n_3 m_+} \hat{X}_+^{m_+} \hat{X}_3^{n_3}, \tag{2.8}$$

$$\hat{X}_-^{n_-} \hat{X}_+^{m_+} = \sum_{i=0}^{\min\{n_-, m_+\}} \lambda^i B_i^{n_-, m_+} \hat{X}_+^{m_+ - i} \hat{X}_3^{2i} \hat{X}_-^{n_- - i}, \tag{2.9}$$

the coefficients $B_i^{n_-, m_+}$ satisfy the recursion relation

$$\begin{aligned} B_0^{n_-, m_+} &= 1, \\ B_i^{n_-, m_+} &= B_i^{n_-, m_+ - 1} + q^{4(m_+ - i)} [[n_- - (i - 1)]]_{q^4} \\ &\quad \times B_{i-1}^{n_-, m_+ - 1}, \end{aligned} \tag{2.10}$$

where $[[n]]_{q^a} \equiv \frac{1 - q^{an}}{1 - q^a}$. As one can see by inserting (2.10) has the solution

$$B_i^{n_-, m_+} = \frac{1}{[[i]]_{q^4}!} \frac{[[n_-]]_{q^4}! [[m_+]]_{q^4}!}{[[n_- - i]]_{q^4}! [[m_+ - i]]_{q^4}!}, \tag{2.11}$$

where $[[n]]_{q^a}! := [[n]]_{q^a} [[n - 1]]_{q^a} \dots [[1]]_{q^a}$, $[[0]]_{q^a}! := 1$. (2.6), (2.7-2.9) and (2.11) together, yield the result

$$\begin{aligned} &W(x_+^{n_+} x_3^{n_3} x_-^{n_-}) W(x_+^{m_+} x_3^{m_3} x_-^{m_-}) \\ &= \sum_{i=0}^{\min\{m_+, n_-\}} C_i^{n_-, m_+} \\ &\quad \times \hat{X}_+^{n_+ + m_+ - i} \hat{X}_3^{n_3 + m_3 + 2i} \hat{X}_-^{n_- + m_- - i} \\ &= W \left(\sum_{i=0}^{\min\{m_+, n_-\}} C_i^{n_-, m_+} \right. \\ &\quad \left. \times x_+^{n_+ + m_+ - i} x_3^{n_3 + m_3 + 2i} x_-^{n_- + m_- - i} \right) \\ &= W((x_+^{n_+} x_3^{n_3} x_-^{n_-}) * (x_+^{m_+} x_3^{m_3} x_-^{m_-})), \end{aligned} \tag{2.12}$$

where $C_i^{n_-, m_+} = \lambda^i q^{2(n_3(m_+ - i) + m_3(n_- - i))} B_i^{n_-, m_+}$. This is the *-product for monomials. In order to obtain the *-product for arbitrary formal power series

$$f = \sum_i a_{i_+, i_3, i_-} x_+^{i_+} x_3^{i_3} x_-^{i_-},$$

we have to substitute

$$q^{n_A} \quad \text{with} \quad q^{\hat{\sigma}^A} = q^{x_A \frac{\partial}{\partial x_A}}, \quad A \in \{3, +, -\} \tag{2.13}$$

(no summation over A) with the usual commutative derivatives.

Applying this substitution to (2.12) we end up at the expression

$$\begin{aligned} f * g &= \sum_{i=0}^{\infty} \lambda^i \frac{x_3^{2i}}{[[i]]_{q^4}!} q^{2(\hat{\sigma}_3 \hat{\sigma}'_+ + \hat{\sigma}_- \hat{\sigma}'_-)} \\ &\quad \times \left(D_{q^4}^- \right)^i f(x) \cdot \left(D_{q^4}^+ \right)^i g(x') \Big|_{x' \rightarrow x}, \end{aligned} \tag{2.14}$$

$f, g \in \mathcal{A}_q$. We have used the q -differentiation operator $D_q^A f(x) = \frac{f(x_A) - f(qx_A)}{x_A - qx_A}$ ([14]) in the above formula.

For practical purposes, we want to know an expansion of expressions (2.12) and (2.14) in the variable $h = \ln q$. One expects that the main contribution to the *-product is made by the expansion coefficients up to h^2 . So that we are not too far away from the classical situation, $h = 0$. For the expression (2.12) we get the expansion

$$\begin{aligned} &(x_+^{n_+} x_3^{n_3} x_-^{n_-}) * (x_+^{m_+} x_3^{m_3} x_-^{m_-}) \\ &= x_+^{n_+ + m_+} x_3^{n_3 + m_3} x_-^{n_- + m_-} \\ &\quad + h \left(a_0^{(1)}(\underline{n}, \underline{m}) x_+^{n_+ + m_+} x_3^{n_3 + m_3} x_-^{n_- + m_-} + \theta(n_-) \theta(m_+) \right) \\ &\quad \times a_1^{(1)}(\underline{n}, \underline{m}) x_+^{n_+ + m_+ - 1} x_3^{n_3 + m_3 + 2} x_-^{n_- + m_- - 1} \\ &\quad + h^2 \left(a_0^{(2)}(\underline{n}, \underline{m}) x_+^{n_+ + m_+} x_3^{n_3 + m_3} x_-^{n_- + m_-} + \theta(n_-) \theta(m_+) \right) \\ &\quad \times a_1^{(2)}(\underline{n}, \underline{m}) x_+^{n_+ + m_+ - 1} x_3^{n_3 + m_3 + 2} x_-^{n_- + m_- - 1} \\ &\quad + \theta(n_- - 1) \theta(m_+ - 1) a_2^{(2)}(\underline{n}, \underline{m}) \\ &\quad \times x_+^{n_+ + m_+ - 2} x_3^{n_3 + m_3 + 4} x_-^{n_- + m_- - 2} \Big) + \mathcal{O}(h^3), \end{aligned} \tag{2.15}$$

where we have the coefficients

$$\begin{aligned} a_0^{(1)}(\underline{n}, \underline{m}) &= 2(n_3 m_+ + m_3 n_-), \\ a_1^{(1)}(\underline{n}, \underline{m}) &= 2n_- m_+, \\ a_0^{(2)}(\underline{n}, \underline{m}) &= 2(n_3 m_+ + m_3 n_-)^2, \\ a_1^{(2)}(\underline{n}, \underline{m}) &= 4n_- m_+ ((n_3 + 1)(m_+ - 1) \\ &\quad + (m_3 + 1)(n_- - 1)), \\ a_2^{(2)}(\underline{n}, \underline{m}) &= 2n_- (n_- - 1) m_+ (m_+ - 1). \end{aligned} \tag{2.16}$$

And in terms of derivatives we find

$$f * g = f(x)g(x) + h \left(2(\hat{\sigma}_3 \hat{\sigma}'_+ + \hat{\sigma}'_- \hat{\sigma}_-) \right)$$

$$\begin{aligned}
 & +2 \frac{x_3^2}{x_+ x_-} \hat{\sigma}_- \hat{\sigma}'_+ \Big|_{x' \rightarrow x} f(x)g(x') \\
 & + h^2 \left(2(\hat{\sigma}_3 \hat{\sigma}'_+ + \hat{\sigma}'_3 \hat{\sigma}_-) \right. \\
 & + 2 \left(\frac{x_3^2}{x_+ x_-} \right)^2 \hat{\sigma}_- (\hat{\sigma}_- - 1) \hat{\sigma}'_+ (\hat{\sigma}'_+ - 1) \\
 & + 4 \frac{x_3^2}{x_+ x_-} \hat{\sigma}_- \hat{\sigma}'_+ ((\hat{\sigma}_3 + 1)(\hat{\sigma}'_+ - 1) \\
 & \left. + (\hat{\sigma}'_3 + 1)(\hat{\sigma}_- - 1)) \right) f(x)g(x') \Big|_{x' \rightarrow x} \\
 & + \mathcal{O}(h^3). \tag{2.17}
 \end{aligned}$$

3 q-deformed 4-dimensional Euclidean space

The procedure to get the *-product for the 4-dimensional Euclidean space is very much the same as in Sect. 2. Therefore we will only state the results. The quantum space algebra is freely generated by the coordinates $\hat{X}_1, \hat{X}_2, \hat{X}_3$ and \hat{X}_4 , divided by the ideal generated by the following relations [9, 15]

$$\begin{aligned}
 \hat{X}_1 \hat{X}_2 &= q \hat{X}_2 \hat{X}_1, \quad \hat{X}_1 \hat{X}_3 = q \hat{X}_3 \hat{X}_1, \\
 \hat{X}_3 \hat{X}_4 &= q \hat{X}_4 \hat{X}_3, \quad \hat{X}_2 \hat{X}_4 = q \hat{X}_4 \hat{X}_2, \\
 \hat{X}_2 \hat{X}_3 &= \hat{X}_3 \hat{X}_2, \quad \hat{X}_4 \hat{X}_1 - \hat{X}_1 \hat{X}_4 = \lambda \hat{X}_2 \hat{X}_3. \tag{3.18}
 \end{aligned}$$

As a basis we use the ordered monomials $\hat{X}_1^{i_1} \hat{X}_2^{i_2} \hat{X}_3^{i_3} \hat{X}_4^{i_4}$, and

$$W(x_1^{i_1} x_2^{i_2} x_3^{i_3} x_4^{i_4}) = \hat{X}_1^{i_1} \hat{X}_2^{i_2} \hat{X}_3^{i_3} \hat{X}_4^{i_4}. \tag{3.19}$$

We get to (2.7), (2.8) and (2.9) analogue formulae

$$\begin{aligned}
 \hat{X}_2^{n_2} \hat{X}_1^{m_1} &= q^{-n_2 m_1} \hat{X}_1^{m_1} \hat{X}_2^{n_2}, \\
 \hat{X}_3^{n_3} \hat{X}_1^{m_1} &= q^{-n_3 m_1} \hat{X}_1^{m_1} \hat{X}_3^{n_3}, \\
 \hat{X}_4^{n_4} \hat{X}_3^{m_3} &= q^{-n_4 m_3} \hat{X}_3^{m_3} \hat{X}_4^{n_4}, \\
 \hat{X}_4^{n_4} \hat{X}_2^{m_2} &= q^{-n_4 m_2} \hat{X}_2^{m_2} \hat{X}_4^{n_4}, \\
 \hat{X}_3^{m_3} \hat{X}_2^{n_2} &= \hat{X}_2^{n_2} \hat{X}_3^{m_3}, \tag{3.20} \\
 \hat{X}_4^{n_4} \hat{X}_1^{m_1} &= \sum_{i=0}^{\min\{n_4, m_1\}} \lambda^i B_i^{n_4, m_1} \hat{X}_1^{m_1-i} \hat{X}_2^i \hat{X}_3^i \hat{X}_4^{n_4-i}
 \end{aligned}$$

where

$$B_i^{n_4, m_1} = \frac{1}{[[i]]_{q^{-2}}! [[n_4 - i]]_{q^{-2}}! [[m_1 - i]]_{q^{-2}}!} \tag{3.21}$$

Therefore the *-product of two monomials has the form

$$\begin{aligned}
 & (x_1^{n_1} x_2^{n_2} x_3^{n_3} x_4^{n_4}) * (x_1^{m_1} x_2^{m_2} x_3^{m_3} x_4^{m_4}) \\
 &= \sum_{i=0}^{\min\{n_4, m_1\}} \lambda^i q^{-(n_2+n_3)(m_1-i) - (m_2+m_3)(n_4-i)} B_i^{n_4, m_1} \\
 & \times x_1^{n_1+m_1-i} x_2^{n_2+m_2+i} x_3^{n_3+m_3+i} x_4^{n_4+m_4-i}. \tag{3.22}
 \end{aligned}$$

Using again the substitution (2.13) we obtain for $f, g \in \mathcal{A}_q$

$$\begin{aligned}
 f * g &= \sum_{i=0}^{\infty} \lambda^i \frac{(x_2 x_3)^i}{[[i]]_{q^{-2}}!} q^{-(\hat{\sigma}_2 + \hat{\sigma}_3) \hat{\sigma}'_1 - (\hat{\sigma}'_2 + \hat{\sigma}'_3) \hat{\sigma}_4} \\
 & \times \left(D_{q^{-2}}^4 \right)^i f(x) \cdot \left(D_{q^{-2}}^1 \right)^i g(x') \Big|_{x' \rightarrow x}, \tag{3.23}
 \end{aligned}$$

with the same definitions and conventions as in the previous section.

Again we want to expand expressions (3.22) and (3.23) in terms of $h = \ln q$. We find

$$\begin{aligned}
 & (x_1^{n_1} x_2^{n_2} x_3^{n_3} x_4^{n_4}) * (x_1^{m_1} x_2^{m_2} x_3^{m_3} x_4^{m_4}) \\
 &= x_1^{n_1+m_1} x_2^{n_2+m_2} x_3^{n_3+m_3} x_4^{n_4+m_4} \\
 & + h \left(a_0^{(1)} x_1^{n_1+m_1} x_2^{n_2+m_2} x_3^{n_3+m_3} x_4^{n_4+m_4} + \theta(n_4) \right. \\
 & \times \theta(m_1) a_1^{(1)} x_1^{n_1+m_1-1} x_2^{n_2+m_2+1} x_3^{n_3+m_3+1} x_4^{n_4+m_4-1} \Big) \\
 & + h^2 \left(a_0^{(2)} x_1^{n_1+m_1} x_2^{n_2+m_2} x_3^{n_3+m_3} x_4^{n_4+m_4} \right. \\
 & + \theta(n_4) \theta(m_1) a_1^{(2)} x_1^{n_1+m_1-1} x_2^{n_2+m_2+1} x_3^{n_3+m_3+1} \\
 & \times x_4^{n_4+m_4-1} + \theta(n_4 - 1) \theta(m_1 - 1) a_2^{(2)} \\
 & \times x_1^{n_1+m_1-2} x_2^{n_2+m_2+2} x_3^{n_3+m_3+2} x_4^{n_4+m_4-2} \Big) \\
 & + \mathcal{O}(h^3), \tag{3.24}
 \end{aligned}$$

where $a_i^{(j)} = a_i^{(j)}(\underline{n}, \underline{m})$,

$$\begin{aligned}
 a_0^{(1)}(\underline{n}, \underline{m}) &= -(n_2 + n_3)m_1 - (m_2 + m_3)n_4, \\
 a_1^{(1)}(\underline{n}, \underline{m}) &= 2n_4 m_1, \\
 a_0^{(2)}(\underline{n}, \underline{m}) &= \frac{1}{2}((n_2 + n_3)m_1 + (m_2 + m_3)n_4)^2, \\
 a_1^{(2)}(\underline{n}, \underline{m}) &= -2n_4 m_1 \left(((n_2 + n_3) + 1)(m_1 - 1) \right. \\
 & \left. + ((m_2 + m_3) + 1)(n_4 - 1) \right), \\
 a_2^{(2)}(\underline{n}, \underline{m}) &= 2n_4(n_4 - 1)m_1(m_1 - 1). \tag{3.25}
 \end{aligned}$$

And in terms of derivatives we find

$$\begin{aligned}
 f * g &= f(x)g(x) + h \left(-(\hat{\sigma}_2 + \hat{\sigma}_3) \hat{\sigma}'_1 \right. \\
 & \left. - (\hat{\sigma}'_2 + \hat{\sigma}'_3) \hat{\sigma}_4 + 2 \frac{x_2 x_3}{x_1 x_4} \hat{\sigma}_4 \hat{\sigma}'_1 \right) f(x)g(x') \Big|_{x' \rightarrow x} \\
 & + h^2 \left(\frac{1}{2}((\hat{\sigma}_2 + \hat{\sigma}_3) \hat{\sigma}'_1 + (\hat{\sigma}'_2 + \hat{\sigma}'_3) \hat{\sigma}_4)^2 \right. \\
 & + 2 \left(\frac{x_2 x_3}{x_1 x_4} \right)^2 \hat{\sigma}_4 (\hat{\sigma}_4 - 1) \hat{\sigma}'_1 (\hat{\sigma}'_1 - 1) \\
 & - 2 \frac{x_2 x_3}{x_1 x_4} \hat{\sigma}_4 \hat{\sigma}'_1 \left(((\hat{\sigma}_2 + \hat{\sigma}_3) + 1)(\hat{\sigma}'_1 - 1) \right. \\
 & \left. \left. + ((\hat{\sigma}'_2 + \hat{\sigma}'_3) + 1)(\hat{\sigma}_4 - 1) \right) \right) f(x)g(x') \Big|_{x' \rightarrow x} \\
 & + \mathcal{O}(h^3). \tag{3.26}
 \end{aligned}$$

The symmetry in all these expressions between x_1 and x_4 , respectively n_4 and m_1 is remarkable. In (3.24) the exponents of the variables x_1 and x_4 are always diminished by the same number. These powers are distributed symmetrically among the coordinates x_2 and x_3 . This stems from the fact that $SO_q(4)$ can be decomposed into 2 independent copies of $SU_q(2)$, as in the classical case. In case of the Lorentz group its decomposition leads also to the tensor product of 2 copies of $SL_q(2)$, which are related to each other via complex conjugation. Thus we will not be able to observe this symmetry between the corresponding Minkowski coordinates, x_0 and x_3 . Additional terms in (3.24) will occur where the powers taken away from x_- and x_+ are not symmetrically distributed among x_0 and x_3 . But still some remnants of the symmetry are present, cf. (4.35).

4 q -deformed Minkowski space

The maybe most important case we want to discuss in this article is a q -deformed version of the Minkowski space, the co-module algebra of the q -deformed Lorentz group [4,5,16–18]. q -Minkowski space is generated by the four coordinates $\hat{X}_0, \hat{X}_+, \hat{X}_3$ and \hat{X}_- , they satisfy the following relations

$$\begin{aligned} \hat{X}_- \hat{X}_0 &= \hat{X}_0 \hat{X}_-, \\ \hat{X}_+ \hat{X}_0 &= \hat{X}_0 \hat{X}_+, \\ \hat{X}_3 \hat{X}_0 &= \hat{X}_0 \hat{X}_3, \\ \hat{X}_- \hat{X}_3 - q^2 \hat{X}_3 \hat{X}_- &= (1 - q^2) \hat{X}_0 \hat{X}_-, \\ \hat{X}_3 \hat{X}_+ - q^2 \hat{X}_+ \hat{X}_3 &= (1 - q^2) \hat{X}_0 \hat{X}_+, \\ \hat{X}_- \hat{X}_+ - \hat{X}_+ \hat{X}_- &= \lambda (\hat{X}_3 \hat{X}_3 - \hat{X}_0 \hat{X}_3). \end{aligned} \tag{4.27}$$

In order to make the calculations easier, we introduce a new set of coordinates $\hat{\tilde{X}}_0, \hat{\tilde{X}}_+, \hat{\tilde{X}}_3, \hat{\tilde{X}}_-$, where

$$\hat{\tilde{X}}_3 \equiv \hat{X}_3 - \hat{X}_0. \tag{4.28}$$

Thus the relations (4.27) become

$$\begin{aligned} \hat{X}_- \hat{\tilde{X}}_3 &= q^2 \hat{\tilde{X}}_3 \hat{X}_-, \quad \hat{\tilde{X}}_3 \hat{X}_+ = q^2 \hat{X}_+ \hat{\tilde{X}}_3, \\ \hat{X}_- \hat{X}_+ - \hat{X}_+ \hat{X}_- &= \lambda (\hat{\tilde{X}}_3 \hat{\tilde{X}}_3 + \hat{X}_0 \hat{\tilde{X}}_3). \end{aligned} \tag{4.29}$$

We again introduce the isomorphism W from the commutative coordinate algebra into the q -deformed Minkowski space

$$W(x_0^{n_0} x_+^{n_+} \tilde{x}_3^{n_3} x_-^{n_-}) = \hat{X}_0^{n_0} \hat{X}_+^{n_+} \hat{\tilde{X}}_3^{n_3} \hat{X}_-^{n_-}, \tag{4.30}$$

the right hand side is defined as our normal ordering.

Using relations (4.29) we get

$$\begin{aligned} \hat{\tilde{X}}_3 \hat{X}_+^{m_+} &= q^{2n_3 m_+} \hat{X}_+^{m_+} \hat{\tilde{X}}_3^{n_3}, \\ \hat{X}_-^{n_-} \hat{\tilde{X}}_3^{m_3} &= q^{2n_- m_3} \hat{\tilde{X}}_3^{m_3} \hat{X}_-^{n_-}, \\ \hat{X}_-^{n_-} \hat{X}_+^{m_+} &= \sum_{i=0}^{\min\{n_-, m_+\}} \lambda^i \hat{X}_+^{m_+ - i} F_i^{n_-, m_+}(\hat{X}_0, \hat{\tilde{X}}_3) \hat{X}_-^{n_- - i}, \end{aligned} \tag{4.31}$$

where the coefficients $F_i^{n_-, m_+}(\hat{X}_0, \hat{\tilde{X}}_3)$ satisfy the recursion relation

$$\begin{aligned} F_i^{n_-, m_+}(\hat{X}_0, \hat{\tilde{X}}_3) &= F_i^{n_-, m_+ - 1}(\hat{X}_0, \hat{\tilde{X}}_3) + F_{i-1}^{n_-, m_+ - 1}(\hat{X}_0, \hat{\tilde{X}}_3) \\ &\quad \times \left(q^{4(m_+ - i)} [[n - (i - 1)]]_{q^4} \hat{\tilde{X}}_3^2 \right. \\ &\quad \left. + q^{2(m_+ - i)} [[n - (i - 1)]]_{q^2} \hat{X}_0 \hat{\tilde{X}}_3 \right), \\ F_0^{n_-, m_+}(\hat{X}_0, \hat{\tilde{X}}_3) &= 1. \end{aligned} \tag{4.32}$$

We could not deduce a closed expression for $F_i^{n_-, m_+}(\hat{X}_0, \hat{\tilde{X}}_3)$ solving the recursion relations (4.32).

However, we can write down what we have so far for the $*$ -product of ordered monomials,

$$\begin{aligned} (x_0^{n_0} x_+^{n_+} \tilde{x}_3^{n_3} x_-^{n_-}) * (x_0^{m_0} x_+^{m_+} \tilde{x}_3^{m_3} x_-^{m_-}) &= \sum_{i=0}^{\min\{n_-, m_+\}} \lambda^i q^{2(n_3(m_+ - i) + m_3(n_- - i))} F_i^{n_-, m_+}(x_0, \tilde{x}_3) \\ &\quad \times x_0^{n_0 + m_0} x_+^{n_+ + m_+ - i} \tilde{x}_3^{n_3 + m_3} x_-^{n_- + m_- - i}. \end{aligned} \tag{4.33}$$

We can rewrite the recursion formula for $F_i^{n_-, m_+}(x_0, \tilde{x}_3)$

$$\begin{aligned} F_j^{n_-, m_+} &= \sum_{i=0}^{m-j} \left(q^{4i} [[n - (j - 1)]]_{q^4} \tilde{x}_3^2 \right. \\ &\quad \left. + q^{2i} [[n - (j - 1)]]_{q^2} x_0 \tilde{x}_3 \right) F_{j-1}^{n_-, m_+}(x_0, \tilde{x}_3)^{(j-1)} \\ &= \sum_{i_0=0}^{m-j} \sum_{i_1=0}^{i_0} \cdots \sum_{i_{j-1}=0}^{i_{j-2}} \prod_{k=0}^{j-1} \sum_{l=0}^{n-(j-k)} \\ &\quad \times \left(q^{4(l+i_k)} \tilde{x}_3^2 + q^{2(l+i_k)} x_0 \tilde{x}_3 \right) \end{aligned} \tag{4.34}$$

and expand this expression in powers of $h = \ln q$. The expansion of $F_i^{n_-, m_+}$ enables us to write down the $*$ -product up to order h^2 . In order to deduce a closed expression we will use the identification of the generators of q -deformed Minkowski space with combinations of the generators of the Drinfeld-Jimbo algebra $\mathcal{U}_q(sl_2)$ [19,20,14].

Expanding expression (4.33) in powers of h reads

$$\begin{aligned} (x_0^{n_0} x_+^{n_+} \tilde{x}_3^{n_3} x_-^{n_-}) * (x_0^{m_0} x_+^{m_+} \tilde{x}_3^{m_3} x_-^{m_-}) &= x_0^{n_0 + m_0} x_+^{n_+ + m_+} \tilde{x}_3^{n_3 + m_3} x_-^{n_- + m_-} \\ &\quad + h \left(a_{0,0}^{(1)}(\underline{n}, \underline{m}) x_0^{n_0 + m_0} x_+^{n_+ + m_+} \tilde{x}_3^{n_3 + m_3} x_-^{n_- + m_-} \right. \\ &\quad \left. + \theta(n_-) \theta(m_+) \sum_{i=0,1} a_{1-i,1+i}^{(1)}(\underline{n}, \underline{m}) \right. \\ &\quad \left. \times x_0^{n_0 + m_0 + (1-i)} x_+^{n_+ + m_+ - 1} \tilde{x}_3^{n_3 + m_3 + (1+i)} x_-^{n_- + m_- - 1} \right) \\ &\quad + h^2 \left(a_{0,0}^{(2)}(\underline{n}, \underline{m}) x_0^{n_0 + m_0} x_+^{n_+ + m_+} \tilde{x}_3^{n_3 + m_3} x_-^{n_- + m_-} \right. \end{aligned}$$

$$\begin{aligned}
& +\theta(n_-)\theta(m_+) \sum_{i=0,1} a_{1-i,1+i}^{(2)}(\underline{n}, \underline{m}) \\
& \times x_0^{n_0+m_0+(1-i)} x_+^{n_++m_+-1} \tilde{x}_3^{n_3+m_3+(1+i)} x_-^{n_-+m_- -1} \\
& +\theta(n_- - 1)\theta(m_+ - 1) \sum_{i=0,1,2} a_{2-i,2+i}^{(2)}(\underline{n}, \underline{m}) \\
& \times x_0^{n_0+m_0+(2-i)} x_+^{n_++m_+-2} \tilde{x}_3^{n_3+m_3+(2+i)} x_-^{n_-+m_- -2} \Big) \\
& +\mathcal{O}(\hbar^3), \tag{4.35}
\end{aligned}$$

where

$$\begin{aligned}
a_{0,0}^{(1)}(\underline{n}, \underline{m}) &= 2(n_3 m_+ + m_3 n_-), \\
a_{1,1}^{(1)}(\underline{n}, \underline{m}) &= a_{0,2}^{(1)}(\underline{n}, \underline{m}) = 2n_- m_+, \\
a_{0,0}^{(2)}(\underline{n}, \underline{m}) &= 2(n_3 m_+ + m_3 n_-)^2, \\
a_{1,1}^{(2)}(\underline{n}, \underline{m}) &= 2n_- m_+ \left((2n_3 + 1)(m_+ - 1) \right. \\
& \quad \left. + (2m_3 + 1)(n_- - 1) \right), \\
a_{0,2}^{(2)}(\underline{n}, \underline{m}) &= 4n_- m_+ \left((n_3 + 1)(m_+ - 1) \right. \\
& \quad \left. + (m_3 + 1)(n_- - 1) \right), \\
a_{2,2}^{(2)}(\underline{n}, \underline{m}) &= \frac{1}{2} a_{1,3}^{(2)}(\underline{n}, \underline{m}) \\
&= a_{0,4}^{(2)}(\underline{n}, \underline{m}) \\
&= 2n_-(n_- - 1)m_+(m_+ - 1). \tag{4.36}
\end{aligned}$$

And in terms of derivatives we find

$$\begin{aligned}
f * g &= f(x)g(x) + \hbar \left(2(\hat{\sigma}_3 \hat{\sigma}'_+ + \hat{\sigma}'_3 \hat{\sigma}_-) \right. \\
& \quad \left. + 2 \frac{\tilde{x}_3^2 + x_0 \tilde{x}_3}{x_+ x_-} \hat{\sigma}_- \hat{\sigma}'_+ \right) f(x)g(x') \Big|_{x' \rightarrow x} \\
& + \hbar^2 \left(2(\hat{\sigma}_3 \hat{\sigma}'_+ + \hat{\sigma}'_3 \hat{\sigma}_-)^2 + 4 \frac{\tilde{x}_3^2}{x_+ x_-} \hat{\sigma}_- \hat{\sigma}'_+ \right. \\
& \quad \times \left((\hat{\sigma}_3 + 1)(\hat{\sigma}'_+ - 1) + (\hat{\sigma}'_3 + 1)(\hat{\sigma}_- - 1) \right) \\
& + 2 \frac{x_0 \tilde{x}_3}{x_+ x_-} \hat{\sigma}_- \hat{\sigma}'_+ \left((2\hat{\sigma}_3 + 1)(\hat{\sigma}'_+ - 1) \right. \\
& \quad \left. + (2\hat{\sigma}'_3 + 1)(\hat{\sigma}_- - 1) \right) + 2 \left(\frac{\tilde{x}_3^2 + x_0 \tilde{x}_3}{x_+ x_-} \right)^2 \\
& \quad \times \hat{\sigma}_- (\hat{\sigma}_- - 1) \hat{\sigma}'_+ (\hat{\sigma}'_+ - 1) \Big) f(x)g(x') \Big|_{x' \rightarrow x} \\
& + \mathcal{O}(\hbar^3). \tag{4.37}
\end{aligned}$$

Finally, we want to deduce a closed expression for the *-product (4.33). To this aim we have a look at the algebra $\mathcal{U}_q(sl_2)$ [14]. The algebra is generated by the four generators E, F, K, K^{-1} , satisfying the relations

$$\begin{aligned}
KE &= q^2 EK, & KF &= q^{-2} FK, & KK^{-1} &= K^{-1}K = 1, \\
EF - FE &= \frac{K - K^{-1}}{q - q^{-1}}. \tag{4.38}
\end{aligned}$$

Further we have [14]

$$\begin{aligned}
F^n E^m &= E^m F^n + \sum_{i=1}^{\min\{n,m\}} (-\lambda)^{-i} \frac{[n]! [m]!}{[i]! [n-i]! [m-i]!} \\
& \times \left(\prod_{j=0}^{i-1} K q^{n-m+j} - K^{-1} q^{-n+m-j} \right) E^{m-i} F^{n-i}, \tag{4.39}
\end{aligned}$$

where $[a] = \frac{q^a - q^{-a}}{q - q^{-1}}$.

The operators L_A, W defined in (4.40) can be interpreted as q -angular momentum operators [4]. They span a proper subalgebra of $\mathcal{U}_q(su_2)$.

$$\begin{aligned}
L_+ &\equiv q^{-3}[2]^{-1/2} E, \\
L_- &\equiv -q^{-2}[2]^{-1/2} KF, \\
L_3 &\equiv q^{-3}[2]^{-1} (qFE - q^{-1}EF), \\
W &\equiv K + q^3 \lambda L_3. \tag{4.40}
\end{aligned}$$

Because of (4.38), these generators satisfy the following relations

$$\begin{aligned}
L_3 L_+ - q^2 L_+ L_3 &= -\frac{W}{q^2} L_+, \\
L_- L_3 - q^2 L_3 L_- &= -\frac{W}{q^2} L_-, \\
L_- L_+ - L_+ L_- &= -\frac{W}{q^3} L_3 + \lambda L_3 L_3, \tag{4.41}
\end{aligned}$$

$$1 = W^2 - q^6 \lambda^2 (L_3 L_3 - q L_+ L_- - q^{-1} L_- L_+).$$

With the substitution

$$W \rightarrow q^3 \lambda \hat{X}_0, \quad L_A \rightarrow \hat{X}_A, \quad A \in \{+, 3, -\}, \quad \mathbf{1} \rightarrow q^6 \lambda^2 \hat{r}^2$$

we regain the relations of q -Minkowski coordinates (4.27) [21]. Now we return to the third equation of (4.31). Using (4.39) one gets

$$\begin{aligned}
\hat{X}_-^n \hat{X}_+^m &= q^{2nm} \hat{X}_+^m \hat{X}_-^n + \sum_{i=1}^{\min\{n,m\}} \frac{[n]! [m]!}{[i]! [n-i]! [m-i]!} \\
& \times \left(\frac{\lambda_-}{\lambda_+} \right)^i q^{2nm+i^2-2im} \\
& \times \left(\prod_{k=0}^{i-1} q^{n-m+k} \hat{X}_3^2 - q^{-n+m-k} \hat{r}^2 \right) \hat{X}_+^{m-i} \hat{X}_-^{n-i}, \tag{4.42}
\end{aligned}$$

where $\hat{r}^2 = -q^{-2} \hat{X}_3^2 - (1 + q^{-2}) \hat{X}_0 \hat{X}_3 + (q + q^{-1}) \hat{X}_+ \hat{X}_-$, and $\lambda_{\pm} = q \pm q^{-1}$. The right hand side of (4.42) still has to be ordered according to the normal ordering. Note that \hat{X}_3^2 and \hat{r}^2 commute, therefore we find

$$q^{i^2-2im} \prod_{k=0}^{i-1} \left(q^{n-m+k} \hat{X}_3^2 - q^{-n+m-k} \hat{r}^2 \right) \hat{X}_+^{m-i} \hat{X}_-^{n-i}$$

$$\begin{aligned}
 &= q^{-i^2} \sum_{k=0}^i (-1)^k q^{(1/2i-k)(i-1)} \begin{bmatrix} i \\ k \end{bmatrix}_q \left(q^{i-2k} \hat{X}_+ \right)^{m-i} \\
 &\times \hat{X}_3^{2(i-k)} \hat{r}^{2k} \left(q^{i-2k} \hat{X}_- \right)^{n-i}, \tag{4.43}
 \end{aligned}$$

where $\begin{bmatrix} i \\ k \end{bmatrix}_q = \frac{[i]!}{[k]! [k-i]!}$. One can also calculate

$$W^{-1} \left(\hat{X}_3^{2(i-k)} \hat{r}^{2k} \right)$$

the last missing link to write down the *-product for q-deformed Minkowski space, and after a lengthy calculation one gets

$$\begin{aligned}
 &W^{-1} \left(\hat{X}_3^{2(i-k)} \hat{r}^{2k} \right) \tag{4.44} \\
 &= \tilde{x}_3^{2(i-k)} \sum_{p=0}^k (q^{4(i-k)} \lambda_+ x_+ x_-)^p S_{k,p}(x_0, \tilde{x}_3),
 \end{aligned}$$

where

$$\begin{aligned}
 &S_{k,p}(x_0, \tilde{x}_3) \\
 &= \begin{cases} 1, & \text{if } p = k \\ \sum_{j_1=0}^p \sum_{j_2=0}^{j_1} \dots \sum_{j_{k-p}=0}^{j_{k-p-1}} \prod_{l=1}^{k-p} a(x_0, q^{2j_l} \tilde{x}_3), & \text{if } 0 \leq p < k \end{cases}, \tag{4.45} \\
 &a(x_0, \tilde{x}_3) = -q^{-2} \tilde{x}_3^2 - (1 + q^{-2}) x_0 \tilde{x}_3.
 \end{aligned}$$

Equations (4.42), (4.44) and (4.45) enable us to order any two monomials in the q-Minkowski generators and to write down the *-product for q-deformed Minkowski space in a closed expression,

$$\begin{aligned}
 f * g &= \sum_{i=0}^{\infty} \left(\frac{\lambda_-}{\lambda_+} \right)^i \sum_{k+j=i} \frac{R_{k,j}(\underline{x})}{[[k]]_{q^2}! [[j]]_{q^2}!} \\
 &\times q^{(2\hat{\sigma}_3 + \hat{\sigma}_- + i)\hat{\sigma}'_+ + (2\hat{\sigma}'_3 + \hat{\sigma}'_+ + i)\hat{\sigma}_-} \\
 &\times \left[(D_{q^2}^-)^i f \right] (x_0, x_+, \tilde{x}_3, q^{j-k} x_-) \\
 &\cdot \left[(D_{q^2}^+)^i g \right] (x'_0, q^{j-k} x'_+, \tilde{x}'_3, x'_-) \Big|_{x' \rightarrow x}, \tag{4.46}
 \end{aligned}$$

with the polynomials

$$\begin{aligned}
 &R_{k,j}(x_0, x_+, \tilde{x}_3, x_-) \\
 &= (-q)^k (q^j \tilde{x}_3^2)^j \sum_{p=0}^k S_{k,p}(x_0, \tilde{x}_3) \lambda_+^p (q^{4j} x_+ x_-)^p \\
 &= W^{-1} \left(\left(q^j \hat{X}_3^2 \right)^j (-q \hat{r}^2)^k \right). \tag{4.47}
 \end{aligned}$$

So finally, we have found both, the expansion of the *-product in powers of h (4.37) and a closed expression (4.46).

5 Remarks

Let us end with a few comments on (2.14), (3.23) and (4.46). First of all, we can see that the formulas for the *-product have a similiar structure in all three cases. The commutative product is modified by an infinite sum of corrections,

$$f * g = fg + \sum_{i=1}^{\infty} h^i B_i(f, g), \tag{5.48}$$

cf. [22]. The i^{th} term is of order $\mathcal{O}(\lambda^i) = \mathcal{O}(h^i)$.

Additionally, there are some kind of mixed scaling operators of the form $q^{a\hat{\sigma}'\hat{\sigma}}$, which lead to a displacement effect. The derivatives in the exponent will shift the argument of the function, such that the value of the *-product at a given point depends not only on their values at that single point. The displacement effect is present in all dimensions and shows that non-commutativity induced by q-deformation implies some kind of non-locality. Especially in Minkowski space, one is forced to reinterpret the concept of causality, as the *-product, which can be considered as some kind of interaction, does not only depend on the nearby past but also on the nearby future.

The remaining operators and factors are responsible for an effect we have already mentioned at the end of Sect. 3. This substitution effect is absent in less than 3 dimensions. It transforms the (plane) coordinates X_+ and X_- (X_1 and X_4 resp.) into the transverse coordinate X_3 and the time coordinate X_0 (X_2 and X_3 resp.). It also shows that physical quantities like charge densities initially restricted to a plane may expand in transverse directions or undergo a mysterious evolution in time.

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